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AN EFFICIENT COMPUTATIONAL PROCEDURE FOR SOLVING THE MULTI-FACILITY RECTILINEAR FACILITIES LOCATION PROBLEM

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ABSTRACT

The multi-facility location problem with rectilinear distances is considered. A necessary and sufficient optimality condition is stated and proved. An algorithm is developed and computational results are given. A new lower bound for multi-facility problems with ℓ_p distances is also given.

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*Visiting Professor in the Department of Management Sciences, University of Waterloo, Waterloo, Ontario in 1976-77.

AN EFFICIENT COMPUTATIONAL PROCEDURE FOR SOLVING THE MULTI-FACILITY RECTILINEAR FACILITIES LOCATION PROBLEM†

Henrik Juel and Robert Love*

INTRODUCTION

A problem that occurs frequently in the area of facilities location concerns the optimal location of several new facilities in relation to a set of existing facilities. The problem that is addressed here is the one in which distances can be considered to be rectangular. A typical situation occurs when one or more new machines must be located on a plant floor on which there are existing machines and storage areas. The floor is arranged in a rectangular grid such that material movement must take place parallel to two orthogonal axes. The objective is to minimize the total material flow costs between the new machines and the existing machines and storage areas. Other applications include the location of service facilities inside office buildings, the location of factories, warehouses and postal stations in urban areas and the design of piping and wiring circuits. Further discussions of applications are found in reference 9.

A practical application involving the location of two new facilities in relation to three existing manufacturing departments is given by Love and Yerex.[5] For this relatively small problem it was possible to use linear programming as the solution technique using a formulation developed by Wesolowsky and Love[9] and Cabot et. al.[1]

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For larger problems, however, the linear programming formulation of the location problem may give rise to extremely large numbers of variables and equations. As an example, consider a problem with $m=350$ existing facilities and $n=70$ new facilities. The linear programming primal has $2mn + n^2 = 53,900$ variables and $mn + n(n-1)/2 = 26,915$ constraints. Even the more efficient linear programming dual has 26,915 bounded variables and 70 constraints. The direct search approach of Pritsker and Ghare[6] is more appealing from a computational point of view, since much larger problems can be solved than with the linear programming formulation. However, Rao[7] and Wesolowsky[8] have provided counter-examples proving that the Pritsker and Ghare procedure may not reach optimality in all cases. The present work may be viewed as an optimizing implementation of their approach.

In this paper we state and prove necessary and sufficient conditions for optimality in all cases. (These conditions were indicated by Rao using the theory of linear programming.) A solution algorithm is formulated incorporating the conditions, and computation results are given. These indicate that the algorithm is very rapid when large numbers of new facilities are not located at a single point. We also give a new lower bound on the optimal solution to multi-facility problems. This bound is generalized to account for ℓ_p distances which are discussed by Love,[3] and Love and Morris.[4]

PROBLEM AND NOTATION

The multi-facility rectilinear facilities location problem in one dimension can be formulated as:

$$\text{minimize } f(x) = \sum_{j=1}^{n} \sum_{i=1}^{m} w_{1ij}|x_j - a_i| + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} w_{2ij}|x_i - x_j|,$$

where $x = (x_1, x_2, \ldots, x_n)$, m is the number of existing facilities, and n is the number of new facilities. w_{1ij} is the non-negative weight between existing facility i and new facility j, for $1 \leq i \leq m$, $1 \leq j \leq n$. w_{2ij} is the non-negative weight between new facilities i and j, for $1 \leq i < j \leq n$. a_i is the (one-dimensional) location of existing facility i, for $1 \leq i \leq m$. x_j is the (one-dimensional) location of new facility j, to be determined, for $1 \leq j \leq n$. It suffices to consider the problem in one dimension, since an unconstrained problem in more dimensions can be decomposed into one-dimensional problems[9]. It is convenient to assume $a_1 < a_2 < \ldots < a_m$ and to define $w_{2ji} = w_{2ij}$ for $1 \leq i < j \leq n$. Let us denote by a current solution a situation in which each new facility coincides with some existing facility. It suffices to consider current solutions, since an optimal solution exists which is a current solution.[1]

For a current solution $x = (x_1, x_2, \ldots, x_n)$, let s_k denote the index-set of new facilities currently coinciding with existing facility k, that is: $s_k = s_k(x) = \{j : x_j = a_k\}$ for $1 \leq k \leq m$. Finally, let $f'(x;y)$ denote the directional derivative of the convex function f at the point $x = (x_1, x_2, \ldots, x_n)$ in the direction $y = (y_1, y_2, \ldots, y_n)$, that is, $f'(x;y) = \lim_{h \to 0+} \frac{f(x+hy) - f(x)}{h}$.

For the objective function of the multi-facility rectilinear facilities location problem, at a current solution x, we obtain:[2]

$$f'(x;y) = \sum_{k=1}^{m} \left\{ \sum_{j \in s_k} b_j y_j + \sum_{j \in s_k} \sum_{\substack{i \in s_k \\ i < j}} w_{2ij} |y_i - y_j| + \sum_{j \in s_k} w_{1kj} |y_j| \right\},$$

where b_j denotes the force exerted on new facility j by the facilities not coinciding with it, that is,

$$b_j = b_j(x) = \sum_{\substack{i=1 \\ i \neq k}}^{m} w_{1ij} \operatorname{sign}(x_j - a_i) + \sum_{\substack{i=1 \\ i \notin s_k}}^{n} w_{2ij} \operatorname{sign}(x_j - x_i) \quad \text{for } j \in s_k,\ 1 \leq k \leq m.$$

Since the objective function is piece-wise linear, the directional derivative is just the increase in the value of the objective function caused by a sufficiently small perturbation of the current solution. That is, $f'(x;y) = f(x + y) - f(x)$, provided that the components of y are small enough in magnitude that the restriction of f to the line segment connecting x and $x + y$ is linear.

NECESSARY AND SUFFICIENT CONDITIONS FOR OPTIMUM

Since the objective function f is convex, a necessary and sufficient condition for a current solution x to be an optimal solution is that the directional derivative $f'(x;y)$ be non-negative for all directions y. This condition is not particularly operational, and to transform it into a useful result, we need the following:

Fact:

$$g_k(y) = \sum_{j\varepsilon s_k} b_j y_j + \sum_{j\varepsilon s_k} \sum_{\substack{i\varepsilon s_k \\ i<j}} w_{2ij}|y_i - y_j| + \sum_{j\varepsilon s_k} w_{1kj}|y_j| \geq 0 \text{ for all } y=(y_1, y_2, \ldots, y_n)$$

if and only if, for all subsets A_k of s_k: $\left|\sum_{j\varepsilon A_k} b_j\right| \leq \sum_{j\varepsilon A_k} \sum_{i\notin A_k} w_{2ij} + \sum_{j\varepsilon A_k} w_{1kj}$

($i\notin A_k$ means $i\varepsilon s_k - A_k$).

Proof:

The only-if-part follows by setting $y_j = z$ for $j\varepsilon A_k$ and $y_j = 0$ for $j\notin A_k$.

For the if-part we need 2 intermediate results. First suppose $z = \min\{y_j : j\varepsilon s_k\} < 0$. Let $A_k = \{j\varepsilon s_k : y_j = z\}$. Let c be any real number satisfying $z + c \leq 0$ and $z + c \leq y_j$ for $j\notin A_k$. Let the vector y' be given by $y_j' = y_j + c$ for $j\varepsilon A_k$ and $y_j' = y_j$ for $j\notin A_k$. Then by direct computation:

$$g_k(y) = g_k(y') + c\left(-\sum_{j\varepsilon A_k} b_j + \sum_{j\varepsilon A_k} \sum_{i\notin A_k} w_{2ij} + \sum_{j\varepsilon A_k} w_{1kj}\right).$$

Next suppose $z = \max\{y_j : j \epsilon s_k\} > 0$. Let $A_k = \{j \epsilon s_k : y_j = z\}$. Let c be any real number satisfying $z-c \geq 0$ and $z-c \geq y_j$ for $j \notin A_k$. Let the vector y' be given by $y'_j = y_j - c$ for $j \epsilon A_k$ and $y'_j = y_j$ for $j \notin A_k$. Then by direct computation

$$g_k(y) = g_k(y') + c\left(\sum_{j\epsilon A_k} b_j + \sum_{j\epsilon A_k} \sum_{i \notin A_k} w_{2ij} + \sum_{j\epsilon A_k} w_{1kj}\right).$$

Now, for any given y we can apply these results repeatedly with $c > 0$, together with the assumption of the if-part, obtaining $g_k(y) = g_k(0)$ + non-negative terms ≥ 0. This fact, in conjunction with the previously given expression for the directional derivative, proves the following.

Result:

A current solution x is an optimal solution if and only if for all subsets A_k of $s_k(x)$: $\left|\sum_{j\epsilon A_k} b_j(x)\right| \leq \sum_{j\epsilon A_k} \sum_{i\epsilon s_k(x)-A_k} w_{2ij} + \sum_{j\epsilon A_k} w_{1kj}$, for $1 \leq k \leq m$.

For use in the algorithm, 2 special cases of the result are given.

1. If the subset $A_k = \{j\}$, then the condition is:

   $$|b_j(x)| \leq \sum_{\substack{i\epsilon s_k(x)\\ i \neq j}} w_{2ij} + w_{1kj}.$$

   If this condition is not satisfied, an improved current solution is obtained by moving the single new facility j from a_k to $a_{k-\text{sign}(b_j(x))}$.

2. If the subset $A_k = s_k(x)$, then the condition is:

   $$\left|\sum_{j\epsilon s_k(x)} b_j(x)\right| \leq \sum_{j\epsilon s_k(x)} w_{1kj}.$$

   If this condition is not satisfied, an improved current solution is obtained by moving the whole cluster of new facilities in $s_k(x)$ from a_k to $a_{k-\text{sign}\left(\sum_{j\epsilon s_k(x)} b_j(x)\right)}$.

ALGORITHM

On the basis of the preceding section, the following algorithm is proposed:

1. Find a current solution.

2. Improve the current solution by moving new facilities one at a time, until no further improvements are possible (using special case 1 of Result).

3. Improve the current solution by moving whole clusters of coinciding new facilities, one whole cluster at a time, until no further improvements are possible (using special case 2 of Result).

4. Check all subsets of each cluster of coinciding new facilities (using Result). If moving some subset improves the current solution, make that move and restart step 4. Otherwise, an optimal solution has been found.

This algorithm has been implemented much like the one by Pritsker and Ghare. Thus, the initial current solution is found by solving n single-facility problems, ignoring the weights between pairs of new facilities. Any move consists of moving one or more new facilities from the current location to the location of a neighboring existing facility. After each move the b_j's are updated, rather than recomputed from the basic data.

COMPUTATIONAL RESULTS

The algorithm was programmed in Fortran, and some test problems were run on a Univac 1110 computer. The computation times reported here represent CPU time exclusive of compilation and reading or generation of input data. The 2 test problems given by Pritsker and Ghare took .05 seconds each. Eight test problems with $m=50$ and $n=10$ were generated, the weights being random numbers between 0 and 99. These problems took an average of .9 seconds each. Most of the time was spent in step 4, since the typical optimal solution involved most or all of the 10

new facilities forming a cluster on or near a_{25}. Of the 8 problems, in 6 cases the optimal solution was found at step 2, in 1 case at step 3 and in 1 case at step 4. For the typical problem, step 4 just performed the optimality test as intended.

Test problems with m=350 and n=70 were generated with biased weights to avoid the excessive and probably atypical clustering of new facilities. The weights were random numbers between 0 and 9, except that w_{1ij} for $1 \leq j \leq 70$ and $5(j-1) + 1 \leq i \leq 5j$ were random numbers between 0 and W-1. Fourteen problems with W=300 took on the average 1.2 seconds each. In one case the optimal solution was found at step 2, in 4 cases at step 3, and in 9 cases at step 4. Fourteen problems with W=400 took on the average 1.1 seconds each. In no cases was the optimal solution found at step 2, in 9 cases at step 3 and in 5 cases at step 4. In these 28 problems, most of the time was spent in step 2. In some problems with a smaller value of W, an excessive amount of time was spent in step 4, and the program was terminated before completion. For example, in a problem with W=200, the solution after step 3 involved a cluster of 21 new facilities. Probably more than 20 minutes would have been spent just to check all the subsets of this cluster. (The computer can check roughly 1000 subsets per second.)

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APPENDIX - PROPERTIES OF THE ℓ_p FUNCTION

Let $F_p(x)$ denote the objective function of a multi-facility location problem with ℓ_p distances in d dimensions:

$$F_p(x) = \sum_{j=1}^{n} \sum_{i=1}^{m} w_{1ij}\, \ell_p(x_j - a_i) + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} w_{2ij}\, \ell_p(x_i - x_j)$$

where $x = (x_1, x_2, \ldots, x_n)$, $x_j = (x_{j1}, x_{j2}, \ldots, x_{jd})$ for $1 \le j \le n$, $a_i = (a_{i1}, a_{i2}, \ldots, a_{id})$ for $1 \le i \le m$ and $\ell_p(z) = \ell_p(z_1, z_2, \ldots, z_d) = \left(\sum_{k=1}^{d} |z_k|^p\right)^{1/p}$.

Let h_k denote the minimal value of the corresponding kth one-dimensional subproblem:

$$h_k = \min\{ \sum_{j=1}^{n} \sum_{i=1}^{m} w_{1ij}|x_{jk} - a_{ik}| + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} w_{2ij}|x_{ik} - x_{jk}|\},\ 1 \le k \le d.$$

Then a lower bound on the optimal value of the d-dimensional problem is given by the ℓ_p-norm of the vector $h = (h_1, h_2, \ldots, h_d)$: $\min\{F_p(x)\} \ge \ell_p(h)$. This generalizes a result by Pritsker and Ghare (6), who consider the case p=2, d=2. The result follows from the triangle inequality for ℓ_p-norms.

$$\left(\sum_{k=1}^{d} |h_k|^p\right)^{1/p} = \left[\sum_{k=1}^{d} \left(\min\left\{\sum_{j=1}^{n} \sum_{i=1}^{m} w_{1ij}|x_{jk} - a_{ik}| + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} w_{2ij}|x_{ik} - x_{jk}|\right\}\right)^p\right]^{1/p}$$

$$\le \left[\sum_{k=1}^{d} \left(\sum_{j=1}^{n} \sum_{i=1}^{m} w_{1ij}|x_{jk} - a_{ik}| + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} w_{2ij}|x_{ik} - x_{jk}|\right)^p\right]^{1/p}$$

$$\le \sum_{j=1}^{n} \sum_{i=1}^{m} w_{1ij}\left(\sum_{k=1}^{d} |x_{jk} - a_{ik}|^p\right)^{1/p} + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} w_{2ij}\left(\sum_{k=1}^{d} |x_{ik} - x_{jk}|^p\right)^{1/p} = F_p(x).$$

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